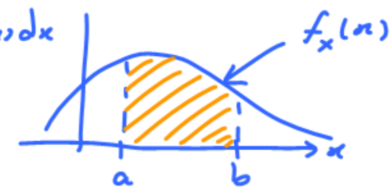
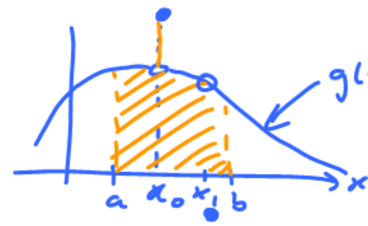
10.2 Properties of PDF and CDF for Continuous Random Variables

10.18. The pdf f_X is determined only almost everywhere[42]. That is, given a pdf f for a random variable X, if we construct a function g by changing the function f at a countable number of points[43], then g can also serve as a pdf for X.

This is because f_X is defined via its integration property. Changing the value of a function at a few points does not change its area under the curve (from a to b)

$P[a < X < b] = \int_a^b f_X(x)dx$





$g(x) = \begin{cases} f(x), & x \neq x_0, x_1 \\ 3, & x = x_0. \\ -2, & x = x_1 \end{cases}$

10.19. The cdf of any kind of random variable X is defined as

$$F_X(x) = P[X \le x].$$

Note that even though there are more than one valid pdfs for any given random variable, the cdf is unique. There is only one cdf for each random variable.

10.20. For continuous random variable, the cdf is continuous.

pdf → cdf

10.21. For continuous random variable, given the pdf $f_X(x)$, we can find the cdf of X by

$F_X(5) = P[X \le 5] = \int_{-\infty}^{5} f_X(x)dx$

$$F_X(x) = P[X \le x] = \int_{-\infty}^{x} f_X(t)dt.$$

cdf → pdf

10.22. Given the cdf $F_X(x)$, we can find the pdf $f_X(x)$ by

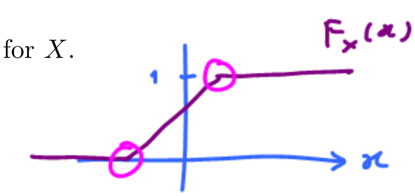
step 1
- If F_X is differentiable at x, we will set

$$\frac{d}{dx}F_X(x) = f_X(x).$$

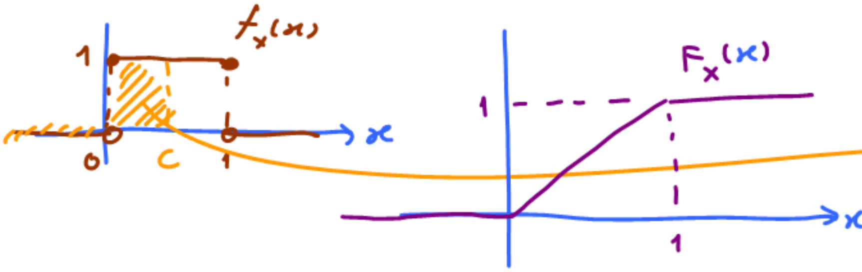
step 2
- If F_X is not differentiable at x, we can set the values of $f_X(x)$ to be any value. Usually, the values are selected to give simple expression. (In many cases, they are simply set to 0.)

[42] Lebesgue-a.e, to be exact

[43] More specifically, if $g = f$ Lebesgue-a.e., then g is also a pdf for X.



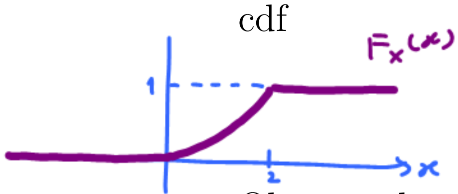
Example 10.23. For the random variable generated by the `rand` command in `MATLAB` or the `rand()` command in `Excel`,



$F_X(x) = P[X \le x]$

$F_X(c) = P[X \le c] = \int_{-\infty}^{c} f_X(x)\,dx = \begin{cases} 0, & c < 0 \\ c, & 0 < c \le 1 \\ 1, & c > 1 \end{cases}$

Example 10.24. Suppose that the lifetime X of a device has the cdf



$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}x^2, & 0 \le x \le 2 \\ 1, & x > 2 \end{cases}$$

$\xrightarrow{\frac{d}{dx}} f_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}x, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$

Observe that it is differentiable at each point x except at $x = 2$. The probability density function is obtained by differentiation of the cdf which gives

We haven't checked $\frac{d}{dx}$ @ $x=0$, $x=2$.

$$f_X(x) = \begin{cases} \frac{1}{2}x, & 0 < x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

But they are only two points on the graph of pdf so, we can define the pdf to be any value there.

At $x = 2$ where F_X has no derivative, it does not matter what values we give to f_X. Here, we set it to be 0.

"we" choose the value to be "0".

10.25. In many situations when you are asked to find pdf, it may be easier to find cdf first and then differentiate it to get pdf.

Exercise 10.26. A point is "picked at random" in the inside of a circular disk with radius r. Let the random variable X denote the distance from the center of the disk to this point. Find $f_X(x)$.

10.27. Unlike the cdf of a discrete random variable, the cdf of a continuous random variable has no jump and is continuous everywhere.

10.28. $p_X(x) = P[X = x] = P[x \le X \le x] = \int_x^x f_X(t)dt = 0.$

Again, it makes no sense to speak of the probability that X will take on a pre-specified value. This probability is always zero.

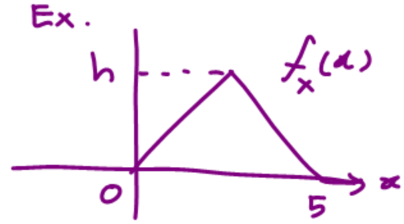
10.29. $P[X = a] = P[X = b] = 0.$ Hence,

$$P[a < X < b] = P[a \le X < b] = P[a < X \le b] = P[a \le X \le b] = F_X(b) - F_X(a)$$

$\int_a^b f_X(x)\,dx = F_X(b) - F_X(a)$

- The corresponding integrals over an interval are not affected by whether or not the endpoints are included or excluded.
- When we work with continuous random variables, it is usually not necessary to be precise about specifying whether or not a range of numbers includes the endpoints. This is quite different from the situation we encounter with discrete random variables where it is critical to carefully examine the type of inequality.

Ex.



$\frac{1}{2} \times h \times 5 = 1$

$h = \frac{2}{5} = 0.4$

10.30. f_X is nonnegative 1 a.e. and $\int_{\cancel{\mathbb{R}}\,-\infty}^{\infty} f_X(x)dx = 1$. 2

Example 10.31. Random variable X has pdf

$$f_X(x) = \begin{cases} ce^{-2x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the constant c and sketch the pdf.

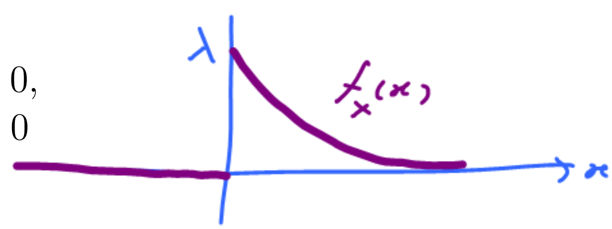
$$\int_{-\infty}^{\infty} f_X(x)\,dx = \cancel{\int_{-\infty}^{0} \underbrace{f_X(x)}_{0}\,dx} + \int_{0}^{\infty} f_X(x)\,dx = \int_{0}^{\infty} ce^{-2x}\,dx = c\,\frac{e^{-2x}}{-2}\Bigg|_{0}^{\infty}$$

$$= \frac{c}{-2}(0-1) = \frac{c}{2} = 1 \quad \Rightarrow \quad c = 2$$

Definition 10.32. A continuous random variable is called ***exponential*** if its pdf is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & x \le 0 \end{cases}$$

for some $\lambda > 0$



Theorem 10.33. Any nonnegative[44] function that integrates to one is a ***probability density function*** (pdf) of some random variable [8, p.139].

[44] or nonnegative a.e.

"Rough" Summary

For continuous RV,

$$\begin{tikzcd}
P[a \leq X \leq b] & & F_X(x) \arrow[ll, "F_X(b) - F_X(a)"'] \arrow[dl, bend right, "\frac{d}{dx} F_X(x)"'] \\
& f_X(x) \arrow[ul, "\int_a^b f_X(x)\,dx"] \arrow[ur, bend right, "\int_{-\infty}^{x} f_X(t)\,dt"'] &
\end{tikzcd}$$

10.3 Expectation and Variance

10.34. ***Expectation***: Suppose X is a continuous random variable with probability density function $f_X(x)$.

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)dx \tag{21}$$

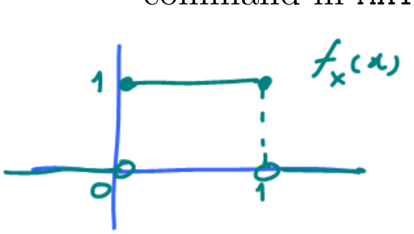
$$\mathbb{E}\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) f_X(x)dx \tag{22}$$

In particular,

$$\mathbb{E}\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)dx$$

$$\text{Var}\, X = \int_{-\infty}^{\infty} (x - \mathbb{E}X)^2 f_X(x)dx = \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2.$$

Example 10.35. For the random variable generated by the `rand` command in `MATLAB` or the `rand()` command in `Excel`,



$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

$$\mathbb{E}\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

$$\text{Var}\, X = \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\sigma_X = \sqrt{\text{Var}\, X} = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}}$$

Example 10.36. For the exponential random variable introduced in Definition 10.32,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^{\infty} x\lambda e^{-\lambda x}dx = \lambda \int_0^{\infty} x e^{-\lambda x}dx = \lambda\left(-\frac{x}{\lambda}e^{-\lambda x} - \frac{1}{\lambda^2}e^{-\lambda x}\right)\Bigg|_0^{\infty}$$

$$= \lambda\left(0 - \left(-\frac{1}{\lambda^2}\right)\right)$$

$$= \frac{1}{\lambda}$$

$$\begin{array}{c|c|c|c} \frac{d}{dx} & & & \int \\ \hline & x & \xrightarrow{+} & e^{-\lambda x} \\ & 1 & \xrightarrow{-} & \frac{1}{-\lambda}e^{-\lambda x} \\ & 0 & \xrightarrow{+} & \frac{1}{(-\lambda)^2}e^{-\lambda x} \end{array}$$

λ = rate   e.g. λ = 2 customers/hour

$\mathbb{E}W = \frac{1}{\lambda}$ = 0.5 hour on average btw. customer.

10.37. If we compare other characteristics of discrete and continuous random variables, we find that with discrete random variables, many facts are expressed as sums. With continuous random variables, the corresponding facts are expressed as integrals.

10.38. All of the properties for the expectation and variance of discrete random variables also work for continuous random variables as well:

(a) Intuition/interpretation of the expected value: As $n \to \infty$, the average of n independent samples of X will approach $\mathbb{E}X$. This observation is known as the "Law of Large Numbers".

(b) For $c \in \mathbb{R}$, $\mathbb{E}[c] = c$

(c) For constants a, b, we have $\mathbb{E}[aX + b] = a\mathbb{E}X + b$.

(d) $\mathbb{E}\left[\sum_{i=1}^{n} c_i g_i(X)\right] = \sum_{i=1}^{n} c_i \mathbb{E}[g_i(X)]$.

(e) $\operatorname{Var} X = \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2$

(f) $\operatorname{Var} X \geq 0$.

(g) $\operatorname{Var} X \leq \mathbb{E}\left[X^2\right]$.

(h) $\operatorname{Var}[aX + b] = a^2 \operatorname{Var} X$.

(i) $\sigma_{aX+b} = |a|\,\sigma_X$.

10.39. ***Chebyshev's Inequality***:

$$P\left[|X - \mathbb{E}X| \geq \alpha\right] \leq \frac{\sigma_X^2}{\alpha^2}$$

or equivalently

$$P\left[|X - \mathbb{E}X| \geq n\sigma_X\right] \leq \frac{1}{n^2}$$

- This inequality use variance to bound the "tail probability" of a random variable.
- Useful only when $\alpha > \sigma_X$

Example 10.40. A circuit is designed to handle a current of 20 mA plus or minus a deviation of less than 5 mA. If the applied current has mean 20 mA and variance 4 $(\text{mA})^2$, use the Chebyshev inequality to bound the probability that the applied current violates the design parameters.

Let X denote the applied current. Then X is within the design parameters if and only if $|X - 20| < 5$. To bound the probability that this does not happen, write

$$P\left[|X-20| \geq 5\right] \leq \frac{\operatorname{Var} X}{5^2} = \frac{4}{25} = 0.16.$$

Hence, the probability of violating the design parameters is at most 16%.

10.41. Interesting applications of expectation:

(a) $f_X(x) = \mathbb{E}\left[\delta\left(X - x\right)\right]$

(b) $P\left[X \in B\right] = \mathbb{E}\left[1_B(X)\right]$

$f_X(c) = \underbrace{\mathbb{E}[\delta(X-c)]}$

$\sum_x g(x) p_X(x)$

$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

$g(x) = \delta(x-c)$

$= \int_{-\infty}^{\infty} \delta(x-c) f_X(x)\, dx$

$= f_X(c)$